

# Duffin-Kemmer-Petiau and Klein-Gordon-Fock Equations for Electromagnetic, Yang-Mills and external Gravitational Field Interactions: proof of equivalence.

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## Abstract

Starting from Generating functional for Green Function (GF), constructed from Lagrangian action in Duffin-Kemmer-Petiau (DKP) theory ( $\mathcal{L}$ -approach) we strictly prove that the physical matrix elements of S-matrix in DKP and Klein-Gordon-Fock (KGF) theories coincide in cases of interaction spin 0 particles with external and quantized Maxwell and Yang-Mills fields and in case of external gravitational field (without or with torsion). For the proof we use reduction formulas of Lehmann Symanzik, Zimmermann (LSZ). We prove that many photons and Yang-Mills particles GF coincide in the both theories, too.

## 1 Introduction

The question on equivalence of DKP and KGF equations has a long history. In 1926 the second order relativistic equation for spin 0 particle has been discovered by O. Klein [1], V. Fock [2], W Gordon [3] and others<sup>1</sup>. After the appearance in

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<sup>1</sup>See [4] where a rich list of references with interesting historical comments can be found, specially about equivalence the DKP and KGF equations. For references see also review [5], where in particular there is references to works I. Gelfand and A. Yaglom, who obtained the first order equation for particle with arbitrary spin, too.

1928 of the famous Dirac equation [6], G. Petiau [7], R. Duffin [8] and N. Kemmer [9] proposed in 1936-39 years the first order DKP equation for description of spin 0 and 1 particles. From the beginning of this time to prove equivalence between DKP and KGF equations many attempts have been undertaken (see reference in [4]). From many questions connected with attempts to proof the equivalence of the both theories we would like to discuss as minimum three important ones: 1. Is there a strict proof of equivalence of the both theories in case of the interaction charged spin 0 particles with eletromagnetic field introduced by minimal way? 2. For which types of interactions is it possible to give strict proof of the equivalence of these both theories? 3. Is there equivalence between these theories on description of process of decay unstable particles (specially decay of  $K_L^0$  mesons)? For details see references in [4].

In QED many processes have been investigated in frame work of DKP theories (see works [10]-[13] and others references in [4]). The main conclusion is that calculation based on DKP and KGF equations yield identical results including one-loop corrections. However the strict proof of equivalence of the both theories for physical matrix elements of  $S$ -matrix does not exist even in QED for any order of the perturbation theory. There was also no attempts to generalize the proof on the other types of interactions. The question about the equivalence of the both theories for description unstable particles is remained open.

The main goal of this paper is to strictly prove that matrix elements of  $S$ -matrix in DKP and KGF theories coincide in cases interaction spin 0 particles with external and quantized Maxwell and Yang-Mills fields and also in case of external gravitational fields (without and with torsion). For the proof we utilized reduction formulas of Lehmann, Symanzik, Zimmermann (LSZ), [14], and Lagrangian approach to construction the generating functional of GF in all the cases of interactions.

As concerned to description of decay unstable particles, we believe that the both theories have to give the same results as well as for all process which are formulated in terms of renormalizable theories. Although this question goes beyond the scope of the paper we give in conclusion the simple example of Lagrangian for description of the decay  $K_L^0$  meson in DKP theory.

The canonical Hamiltonian approach to quantization of DKP equations has been developed in work [15] where it has been proved that generating functional for GF in DKP theory coincides with that of in Lagrangian approach. Thus we start in this paper from generating functional for GF constructed from Lagrangian action with external sources in exponent of functional integral (named  $\mathcal{L}$ -approach). This approach makes simpler the proof of the equivalence of the both theories .

In section 2 the generating functionals are constructed in DKP theory for GF of spin 0 particles interacted with Maxwell and Yang-Mills fields and with external gravitational field (without torsion and with one). We also prove that all many photons GF (not only matrix elements of  $S$ -matrix) coincide on DKP and KGF theories.

In section 3 the equivalence of the matrix elements of  $S$ -matrix in the both

theories are proved for all above mentioned interactions, including non-abelian theories. For strict proof of the results we utilized the wave packets instead of plane waves. In section 4 are given short conclusions about basic results.

## 2 Generating functional

First one apply  $\mathcal{L}$ -approach for construction of the generating functional of GF in DKP theory for charged 0-spin particles interacting with quantized EM field  $A_\mu(x)$ . From general consideration the generating functional in  $\mathcal{L}$ -approach has the following form (in  $\alpha$ -gauge)<sup>2</sup>:

$$Z(\overline{\mathcal{J}}, \mathcal{J}, \mathcal{J}_\mu) = Z_0^{-1} \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\overline{\psi} \exp \left\{ i \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 + \mathcal{J}_\mu A^\mu + \overline{\psi}(x) (i\beta_\mu \mathcal{D}^\mu - m) \psi(x) + \overline{\mathcal{J}}\psi + \overline{\psi}\mathcal{J} \right) \right\} \quad (1)$$

where

$$Z_0 = Z(0, 0, 0); \quad D^\mu = \partial^\mu - ieA^\mu \quad (2)$$

After integration over  $\psi(x)$  and  $\overline{\psi}(x)$  we get:

$$Z(\overline{\mathcal{J}}, \mathcal{J}, \mathcal{J}_\mu) = Z_0^{-1} \int \mathcal{D}A_\mu \exp \left\{ i \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 + Tr \ln S(x, x, A) - \int d^4y \overline{\mathcal{J}}(x) S(x, y, A) \mathcal{J}(y) \right) \right\} \quad (3)$$

here

$$S(x, y, A) = (i\beta_\mu D^\mu - m)^{-1} \delta^4(x - y) \quad (4)$$

is GF of particles in the field  $A_\mu(x)$ ; the term  $Tr \ln S(x, x, A)$  is responsible for arising of the all diagrams of vacuum polarization. This term can be transformed to the following view<sup>3</sup>,

$$\begin{aligned} \det S(x, y, A) &= C \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \exp \left\{ i \int \overline{\psi} (i\beta_\mu D^\mu - m) \psi \right\} \\ &= C \int \Pi_\alpha \mathcal{D}\varphi_\alpha \mathcal{D}\varphi_\alpha^* \exp \left\{ i \int d^4x (-\varphi^* D_\mu \varphi^\mu + \varphi^{*\mu} D_\mu \varphi - m(\varphi^* \varphi + \varphi^{*\mu} \varphi_\mu)) \right\} \\ &= C \int \mathcal{D}\varphi^* \mathcal{D}\varphi \exp \left\{ -\frac{i}{m} \int d^4x \varphi^* (D_\mu D^\mu + m^2) \varphi \right\} \\ &= \det G(x, y, A) = \exp Tr \ln G(x, x, A) \end{aligned} \quad (5)$$

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<sup>2</sup>Notation see in [15]

<sup>3</sup>We utilized the equation (24), which expresses  $\psi_\alpha(x), \overline{\psi}_\alpha(x)$  through the components.

where

$$G(x, y, A) = (D_\mu D^\mu + m^2)^{-1} \delta^4(x - y) \quad (6)$$

is the GF of KGF equation in EM field  $A_\mu(x)$ . Thus utilizing Eqs. (3), (5), (6) we get

$$\begin{aligned} Z(\overline{\mathcal{J}}, \mathcal{J}, \mathcal{J}_\mu) &= Z_0^{-1} \int \mathcal{D}A_\mu \exp \left\{ i \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \right. \right. \\ &\quad \left. \left. + \mathcal{J}_\mu A^\mu + Tr \ln G(x, x, A) - \int d^4y \overline{\mathcal{J}}(x) S(x, y, A) \mathcal{J}(y) \right) \right\} \end{aligned} \quad (7)$$

From Eqs. (6), (7) we get to important conclusions: The all many photons GF (Not only matrix elements of  $S$ -matrix for real photons) coincide in DKP and KGF theories.

In general case one can obtain the generating functional or GF in external  $A_\mu^{ex}$  by substitution in Eqs. (1), (2)

$$D^\mu \rightarrow \partial^\mu - ie(A^\mu + A_{ex}^\mu) \quad (8)$$

However, if  $A_{ex}^\mu$  is not so strong to create pairs of particles, the generating functional in  $A_{ex}^\mu$  will be equal to:

$$Z^{ex} = \exp \left\{ -i \int d^4x d^4y \overline{\mathcal{J}}(x) S(x, y, A^{ex}) \mathcal{J}(y) \right\} \quad (9)$$

and particles do not interact between themselves but only with  $A_{ex}^\mu$ .

Now we consider interaction DKP spin-0 charged particles with external gravitational fields. The Lagrangian density in this case can be written in the form [16]

$$\begin{aligned} \mathcal{L} &= \sqrt{-g} \left\{ \frac{i}{2} \left[ \overline{\psi} \beta^\mu \left( \partial_\mu + \frac{1}{2} \omega_\mu^{ab} S_{ab} \right) \psi \right. \right. \\ &\quad \left. \left. - \left( \partial_\mu \overline{\psi} - \frac{1}{2} \omega_\mu^{ab} \overline{\psi} S_{ab} \right) \beta^\mu \psi - m \overline{\psi} \psi \right] \right\} \end{aligned} \quad (10)$$

Here as in case Dirac equation, we have to introduce nontrivial tetrad field  $e_a^\mu(x)$ , which can be used to define the Riemannian metric ( $a$ -vector Lorentz index;  $\mu$ -riemannian one):

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}, \quad \eta_{ab} = \text{diag}\{1, -1, -1, -1\} \quad (11)$$

with properties:

$$\begin{aligned} e_\mu^a e_a^\nu &= \delta_\mu^\nu, \quad e_\mu^a e^{\mu b} = \eta^{ab} \\ \sqrt{-g} &= \det e_a^\mu = e \end{aligned} \quad (12)$$

$\beta_\mu = e_\mu^a \beta_a$ , where  $5 \times 5$  matrix  $\beta_a$  satisfy usual DKP commutation relations [9]; the connection  $\omega_\mu^{ab}$  equal to (see for instance the book [17]):

$$\omega_\mu^{ab} = e_\lambda^a e^{\nu b} \Gamma_{\mu\nu}^\lambda - e^{\nu b} \partial_\mu e_\nu^a \quad (13)$$

We consider two different cases: without torsion or with one. Without torsion Levi-Civita connection is

$$\Gamma_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda = \overset{\circ}{\Gamma}_{\mu\nu}^\lambda = \frac{1}{2} \overset{\circ}{g}^{\lambda\rho} \left( \partial_\mu \overset{\circ}{g}_{\rho\nu} + \partial_\nu \overset{\circ}{g}_{\rho\mu} - \partial_\rho \overset{\circ}{g}_{\mu\nu} \right) \quad (14)$$

and

$$\partial_\mu \sqrt{-g} = \frac{\sqrt{-g}}{2} g^{\rho\lambda} \partial_\mu g_{\rho\lambda} \equiv \sqrt{-g} \overset{\circ}{\Gamma}_{\rho\mu}^{\rho} \quad (15)$$

If torsion is not equal zero  $\Gamma_{\mu\nu}^\lambda$  can be defined as Cartan connection, [18]:

$$\Gamma_{\mu\nu}^\lambda = e_a^\lambda \partial_\nu e_\mu^a \quad (16)$$

Now from Eqs. (13) and (16) we obtain:

$$\omega_\mu^{ab} = e_\lambda^b e^{\nu a} (\Gamma_{\mu\nu}^\lambda - \Gamma_{\nu\mu}^\lambda) \quad (17)$$

or

$$\omega_{\mu\nu}^\lambda = e_{\nu a} e_b^\lambda \omega_\mu^{ab} = \Gamma_{\mu\nu}^\lambda - \Gamma_{\nu\mu}^\lambda \equiv 2Q_{\mu\rho}^\lambda \quad (18)$$

where the torsion  $Q_{\mu\rho}^\lambda$  is a third rank tensor [17]. Thus we have two different Lagrangian depending on Eqs. (14) or (16) which define the  $\overset{\circ}{\omega}_\mu^{ab}$  or  $\omega_\mu^{ab}$  in Eq. (17). However, in both cases from Eq. (10) we get the form-equivalent equations for  $\psi$  (The equation for  $\psi(x)$  in case Einstein-Cartan gravity [17] is discussed shortly in p.2 in Conclusion):

$$\left( i\beta^\mu \left( \partial_\mu + \frac{1}{2} \omega_\mu^{ab} S_{ab} \right) - m \right) \psi \equiv (i\beta_\mu \nabla^\mu - m) \psi = 0 \quad (19)$$

To show it in case (17) when for instance torsion not equals zero we can transform Eq. (10) (omiting total derivatives) to the following view:

$$\begin{aligned} \mathcal{L} = & \sqrt{-g} \left\{ i\bar{\psi} \left( \beta_\mu \partial^\mu + \frac{1}{2} \omega_\mu^{ab} \beta^\mu S_{ab} - m \right) \psi \right. \\ & \left. + \frac{i}{2} \bar{\psi} \left( \partial_\mu (\beta^\mu \sqrt{-g}) - \omega_\mu^{\mu b} \beta_b \right) \psi \right\} \end{aligned} \quad (20)$$

We utilize at the derivation of Eq. (20) the following equation:

$$\omega_\mu^{ab} S_{ab} \beta^\mu = \omega_\mu^{ab} \beta^\mu S_{ab} - 2\omega_\mu^{\mu a} \beta_a \quad (21)$$

One shows that the last term in Eq. (20) equals zero:

$$\begin{aligned}\partial_\mu (\sqrt{-g}\beta^\mu) &= \beta^a \partial_\mu (e_a^\mu e) = \beta^a [(\partial_\mu e_a^\mu) e + e_a^\mu \partial_\mu e] \\ &= e\beta^a [(\partial_\mu e_a^\mu) + e^\mu \Gamma_{\rho\mu}^\rho] = e\beta^a \omega_{\mu a}^b e_b^\mu = e\omega_{\mu}^{\mu b} \beta_b\end{aligned}\quad (22)$$

where we used that [16, 17]  $(\partial_\mu + \omega_{\mu a}^b - \Gamma_{\lambda\mu}^\lambda) e_a^\mu = 0$ . Thus the last term in Eq. (20) equals zero. We get the same answer without torsion, too. Then Lagrangian (20) can be written so:

$$\mathcal{L} = \sqrt{-g} \{ i\bar{\psi} (\beta_\mu \nabla^\mu - m) \psi \} \quad (23)$$

One writes down the Lagrangian (10),(23) in component form utilizing the expressions for  $\psi_\alpha(x), \bar{\psi}_\alpha(x)$ :

$$\begin{aligned}\psi_\alpha &= (\varphi, \varphi^0, \varphi^1, \varphi^2, \varphi^3, ) \\ \bar{\psi}_\alpha &= (\psi^* \eta)_\alpha = (\varphi^*, \varphi^{*0}, -\varphi^{*1}, -\varphi^{*2}, -\varphi^{*3}, )\end{aligned}\quad (24)$$

we obtain<sup>4</sup>

$$L = \sqrt{-g} \{ -\varphi^* D_\mu \varphi^\mu + \varphi^{*\mu} \partial_\mu \varphi - m (\varphi^2 + \varphi_\mu^* \varphi^\mu) \} \quad (25)$$

where

$$D_\mu = \partial_\mu + \Gamma_{\nu\mu}^\nu \quad (26)$$

From Eqs. (25), (26) we get the following  $\mathcal{L}$ -equations [18]:

$$D_\mu \varphi^\mu + m\varphi = 0, \quad \partial_\mu \varphi + m\varphi_\mu = 0 \quad (27)$$

$$D_\mu \partial^\mu \varphi + m^2 \varphi = 0 \quad (28)$$

The same equations arise for  $\varphi^*$ .

If  $D_\mu = \overset{\circ}{D}_\mu \equiv \partial_\mu + \overset{\circ}{\Gamma}_{\nu\mu}^\nu$  one gets KGF equation without torsion (see Eqs. (18), (19) in [18]); if  $\Gamma_{\mu\nu}^\lambda$  is defined by Eq. (16) we get equation for scalar particles with torsion (see (27), (28) in [18]).

Thus on the classical level it is proved equivalence of DKP and KGF equations in external gravitational fields [16].

On the quantum level ( $2^{nd}$ -quantization) we must construct (also as in external  $A^{ex}$ ) generating functional for GF of spin-0 particles, interacting with external gravitational field  $e_\mu^a(x)$ . By definition the such generating functional in  $\mathcal{L}$ -approach is (see Eq. (23)):

$$Z(\bar{\mathcal{J}}, \mathcal{J}) = Z_0^{-1} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left\{ i \int d^4x \sqrt{-g} (\bar{\psi} (i\beta_\mu \nabla^\mu - m) \psi + \bar{\mathcal{J}}\psi + \bar{\psi}\mathcal{J}) \right\} \quad (29)$$

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<sup>4</sup>For neutral particle  $\mathcal{L} = \frac{1}{2} \sqrt{-g} \{ -\varphi D_\mu \varphi^\mu + \varphi^\mu \partial_\mu \varphi - m^2 (\varphi^2 + \varphi_\mu \varphi^\mu) \}$

where  $Z_0 = Z(0, 0)$ ,  $\overline{\mathcal{J}}$ ,  $\mathcal{J}$  are external currents.

Integrating in Eq. (29) over  $\overline{\psi}$  and  $\psi$  we obtain:

$$Z(\overline{\mathcal{J}}, \mathcal{J}) = \exp \left\{ -i \int d^4x d^4y e(x) e(y) \overline{\mathcal{J}}(x) S(x, y, e) \mathcal{J}(y) \right\} \quad (30)$$

Here we have introduced the total GF of DKP particle in external gravitational field  $e_a^\mu$ :

$$S(x, y, e) = (i\beta_\mu \nabla_x^\mu - m)^{-1} \delta^4(x - y) e^{-1}(y) \quad (31)$$

From Eq. (29) we also have:

$$S(x, y, e) = -i \langle 0 | T \widehat{\psi}(x) \widehat{\overline{\psi}}(y) | 0 \rangle \quad (32)$$

We suppose that external  $e_\mu^a(x)$  is enough weak to create pairs of new particles. Thus one can consider, in this case, only one particle GF in external  $e_\mu^a(x)$ , since each particle interacts with  $e_\mu^a(x)$ , but not between themselves.

### 3 Equivalence between physical matrix elements of S-matrix in DKP and KGF equations

We use LSZ reduction formulas [14] for proof of the equivalence of the both theories. The main goal of LSZ approach is to express the matrix elements of S-matrix through total many particles GF, for instance, Eq. (5), making minimal assumptions as possible. To apply this approach to DKP theory we must to write down the operator's solution of the free DKP equations. Taking into account that in DKP theory there are only two linearly independent solutions [11] of the free equation, one can write:

$$\widehat{\psi}_{in}^{out}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p \left\{ u^-(\mathbf{p}) \widehat{a}_{in}^-(\mathbf{p}) e^{-ipx} + u^+(\mathbf{p}) \widehat{b}_{in}^+(\mathbf{p}) e^{ipx} \right\} \quad (33)$$

$$\widehat{\overline{\psi}}_{in}^{out}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p \left\{ \overline{u}^-(\mathbf{p}) \widehat{a}_{in}^+(\mathbf{p}) e^{ipx} + \overline{u}^+(\mathbf{p}) \widehat{b}_{in}^-(\mathbf{p}) e^{-ipx} \right\} \quad (34)$$

here

$$\begin{aligned} (\widehat{p} \mp m) u^\pm(\mathbf{p}) &= 0, \quad \overline{u}^\mp(\mathbf{p}) (\widehat{p} \pm m) = 0, \quad \widehat{p} = \beta_\mu p^\mu \\ p_0 &= (\mathbf{p}^2 + m^2)^{1/2} \equiv \omega(\mathbf{p}) = \omega \end{aligned} \quad (35)$$

operators  $\widehat{a}_{in}^\pm, \widehat{b}_{in}^\pm$  satisfy to usual commutation relations and the solutions in component form are:

$$u_\alpha^\pm = \sqrt{\frac{m}{2\omega}} \left( 1, \pm \frac{i\omega}{m}, \pm \frac{ip^1}{m}, \pm \frac{ip^2}{m}, \pm \frac{ip^3}{m} \right) \quad (36)$$

It is easy to check the scalar products are:

$$\begin{aligned} u^{*\mp}(\mathbf{p})\beta_0 u^\mp(\mathbf{p}) &= \bar{u}^\mp(\mathbf{p})\beta_0 u^\mp(\mathbf{p}) = \pm 1 \\ \bar{u}^\pm(\mathbf{p})\beta_0 u^\mp(\mathbf{p}) &= 0 \end{aligned} \quad (37)$$

Operators  $\hat{a}_{in, out}^\pm, \hat{b}_{in, out}^\pm$  are that of creation and annihilation with positive and negative charges accordingly.

From Eqs. (33) and (34) one also has:

$$\hat{a}_{in, out}^-(\mathbf{p}) = \frac{1}{(2\pi)^{3/2}} \int d^3x e^{-ipx} \bar{u}^-(\mathbf{p}) \beta_0 \hat{\psi}_{in, out}(x) \quad (38)$$

$$\hat{b}_{in, out}^+(\mathbf{p}) = \frac{1}{(2\pi)^{3/2}} \int d^3x e^{ipx} \bar{u}^+(\mathbf{p}) \beta_0 \hat{\psi}_{in, out}(x) \quad (39)$$

$$\hat{a}_{in, out}^+(\mathbf{p}) = \left( \hat{a}_{in, out}^-(\mathbf{p}) \right)^*, \quad \hat{b}_{in, out}^-(\mathbf{p}) = \left( \hat{b}_{in, out}^+(\mathbf{p}) \right)^* \quad (40)$$

However, decomposition (33), (34) of the operators  $\hat{\psi}_{in, out}(x), \hat{\bar{\psi}}_{in, out}(x)$  over the operators  $\hat{a}_{in, out}^\pm(\mathbf{p}), \hat{b}_{in, out}^\pm(\mathbf{p})$ , which are that of creation and annihilation of the plane wave do not give possibility to construct the total set of the normalizable physical states in Hilbert space. Moreover, utilizing these states for construction reduction formulas of LSZ we can not carry out strictly mathematical proof of the equivalence of DKP and KGF theories. For these aims we have to construct new operators which create and annihilate the wave packets states. We define the new operators in the following decomposition (we omit sign  $\hat{\phantom{x}}$  over operators)

$$a_{in, out}^\mp(\mathbf{p}) = \sum_{n=0}^{\infty} f_n(\mathbf{p}) a_{in, out}^\mp(\mathbf{n}) \quad (41)$$

$$b_{in, out}^\mp(\mathbf{p}) = \sum_{n=0}^{\infty} f_n(\mathbf{p}) b_{in, out}^\mp(\mathbf{n})$$

Here  $f_n(\mathbf{p}) = f_n^*(\mathbf{p})$  are total orthonormalized set of functions with properties<sup>5</sup>

$$\int f_n(\mathbf{p}) f_m(\mathbf{p}) d^3\mathbf{p} = \delta_{nm}, \quad \sum_{n=0}^{\infty} f_n(\mathbf{p}) f_n(\mathbf{q}) = \delta^3(\mathbf{p} - \mathbf{q}) \quad (42)$$

from Eqs.(41) and (42) we get:

$$a_{in, out}^\pm(\mathbf{n}) = \int d^3\mathbf{p} f_n(\mathbf{p}) a_{in, out}^\pm(\mathbf{p}), \quad b_{in, out}^\pm(\mathbf{n}) = \int d^3\mathbf{p} f_n(\mathbf{p}) b_{in, out}^\pm(\mathbf{p}) \quad (43)$$

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<sup>5</sup>We choose  $f_n(\mathbf{p}) = f_n^*(\mathbf{p}) = \exp\left(-\frac{\mathbf{p}^2}{2m^2}\right) \prod_{i=1}^3 H_n\left(\frac{p_i}{m}\right)$ , where  $H_n(x)$  denotes a Hermite polynomial of degree  $n$ .



From commutations relations:

$$\left[ a_{out}^- (\mathbf{p}), a_{in}^+ (\mathbf{q}) \right] = \left[ b_{out}^+ (\mathbf{p}), b_{in}^+ (\mathbf{q}) \right] = \delta^3 (\mathbf{p} - \mathbf{q}) \quad (44)$$

and from Eqs. (41)-(43) one get:

$$\left[ a_{out}^- (n), a_{in}^+ (m) \right] = \left[ b_{out}^- (n), b_{in}^+ (m) \right] = \delta_{mn} \quad (45)$$

Vacuum and  $S$ -matrix are defined as usual

$$a_{out}^- (n) |0\rangle_{out}^{in} = b_{out}^- (n) |0\rangle_{out}^{in} = 0, \quad S |0\rangle_{out}^{in} = |0\rangle_{out}^{in} \quad (46)$$

$$a_{out}^+ = a_{in}^+ S \quad (47)$$

The any physical matrix element of  $S$ -matrix one has:

$$\langle n', m'; out | n, m; in \rangle = \langle n', m'; in | S | n, m; in \rangle \quad (48)$$

were  $n + m = n' + m'$  is conservation of charge and

$$|n, m;_{out}^{in} \rangle \equiv \prod_{i=1}^n \prod_{j=1}^m a_{out}^+ (n_i) b_{out}^+ (n_i) |0\rangle \quad (49)$$

Now we can formulate main assumption of LSZ approach<sup>6</sup>: for any matrix elemnts of Heisenberg operators  $\hat{\psi}(x)$  and  $\hat{\bar{\psi}}(x)$  the following asymptotic relation are implemented:

$$\lim_{x_0 = \mp \infty} \langle n', m';_{out}^{in} | \hat{\psi}(x) | n, m;_{out}^{in} \rangle = \langle n', m';_{out}^{in} | \hat{\psi}_{in}(x) | n, m;_{out}^{in} \rangle \quad (50)$$

and the same relations for  $\hat{\bar{\psi}}(x)$ .

In order to do not complicate the proof of the equivalence of the DKP and KGF theories we restricted by considering of the matrix elements of  $S$ -matrix for particles with the same (positive) charges and one utilizes LSZ reduction formula. We have:

$$\begin{aligned} \langle 0 | \prod_{i=1}^k a_{out}^- (n_i) \prod_{j=1}^k a_{in}^+ (m_j) | 0 \rangle &= \langle 0 | \prod_{i=1}^{k-1} a_{out}^- (n_i) a_{out}^- (n_k) \prod_{j=1}^k a_{in}^+ (m_j) | 0 \rangle \\ &= C \langle n_1, \dots, n_{k-1}; out | \int d^4 x f_{n_k}^{*-} (x) \\ &\quad \left( i \beta_\mu \overrightarrow{\partial}_x^\mu - m \right) \hat{\psi}(x) | n_1, \dots, n_k; in \rangle \end{aligned} \quad (51)$$

Here

$$f_{\alpha; n_k}^{*-} (x) = \frac{1}{(2\pi)^{3/2}} \int d^3 p \bar{u}_\alpha^- (\mathbf{p}) f_{n_k} (\mathbf{p}) e^{ipx}, \quad p_0 = \omega (\mathbf{p}) \quad (52)$$

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<sup>6</sup>This assumption can be proved in case of microcasuality theories, when comutator  $[\psi(x), \bar{\psi}(y)] = 0, \quad (x - y)^2 < 0$

and  $C$ - not essential constant

Utilizing Eqs. (24), (36), (52) we can rewrite Eq. (51) in component form:

$$\begin{aligned} \langle n_1, \dots, n_{k-1}; out | \int d^4x d^3\mathbf{p} f_{n_k}(\mathbf{p}) \left\{ -\frac{1}{m} e^{ipx} \left( \vec{\square}_x + m^2 \right) \widehat{\varphi}(x) \right. \\ \left. + \frac{\partial}{\partial x^\mu} \left[ e^{ipx} \left( \frac{1}{m} \partial_x^\mu \widehat{\varphi}(x) - \widehat{\varphi}^\mu(x) \right) \right] \right\} | n_1, \dots, n_k; in \rangle \end{aligned} \quad (53)$$

The main idea of the proof is to show that the second term under total derivative in Eq. (53) is equal to zero. We have<sup>7</sup>

$$\begin{aligned} \int d^4x \int d^3\mathbf{p} f_{n_k}(\mathbf{p}) \frac{\partial}{\partial x^\mu} \left[ e^{ipx} \left( \frac{1}{m} \partial_x^\mu \widehat{\varphi}(x) - \widehat{\varphi}^\mu(x) \right) \right] \\ = \int d\sigma_\mu \int d^3\mathbf{p} f_{n_k}(\mathbf{p}) e^{ipx} \left( \frac{1}{m} \partial_x^\mu \widehat{\varphi}(x) - \widehat{\varphi}^\mu(x) \right) \end{aligned} \quad (54)$$

One chooses the surface  $\sigma_\mu$  so:

$$\sigma_\mu : \{-T \leq x_0 \leq T; \quad -L \leq x_i \leq L; \quad i = 1, 2, 3\} \quad (55)$$

The first term,  $\mu = 0$  equals

$$\begin{aligned} \int d^3x \left[ \int d^3\mathbf{p} f_{n_k}(\mathbf{p}) e^{i\omega T + i\mathbf{p}\mathbf{x}} \left( \frac{1}{m} \frac{\partial}{\partial T} \widehat{\varphi}(\mathbf{x}, T) - \widehat{\varphi}^0(\mathbf{x}, T) \right) \right. \\ \left. - \int d^3\mathbf{p} f_{n_k}(\mathbf{p}) e^{-i\omega T + i\mathbf{p}\mathbf{x}} \left( \frac{1}{m} \frac{\partial}{\partial T} \widehat{\varphi}(\mathbf{x}, -T) - \widehat{\varphi}^0(\mathbf{x}, -T) \right) \right] \end{aligned} \quad (56)$$

Since

$$\lim_{T \rightarrow \infty} \frac{\partial}{\partial T} \widehat{\varphi}(\mathbf{x}, \pm T) = \frac{\partial}{\partial T} \widehat{\varphi}_{out}^{in}(\mathbf{x}, \pm T) = \widehat{\varphi}_{out}^0(\mathbf{x}, \pm T) \quad (57)$$

the first term (56) disappears in limit  $T \rightarrow \pm\infty$ .

It is enough to consider only one term in Eqs. (54) and (55), for instance  $i = 1$ , and to show that it goes to zero at  $L \rightarrow \pm\infty$  due to Gaussian properties of the packets  $f_n(\mathbf{p})$  (see<sup>5</sup> p.8)

The term  $\mu = 1$  is:

$$\begin{aligned} I(L) = \int dx_0 dx_\perp \int d^3\mathbf{p} f(\mathbf{p}) e^{i\omega x_0 - ip_\perp x_\perp} \left\{ e^{-ip_1 x_1} \left( \frac{1}{m} \partial_x^1 \widehat{\varphi}(x) - \widehat{\varphi}^1(x) \right) \right. \\ \left. - e^{-ip_1 x_1} \left( \frac{1}{m} \partial_x^1 \widehat{\varphi}(x) - \widehat{\varphi}^1(x) \right) \right\}_{x_1=L}^{x_1=-L} \end{aligned} \quad (58)$$

where  $p_i = (p_1, p_\perp)$ ,  $x_i = (x_1, x_\perp)$ ,  $f_n = f_{n_k}$

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<sup>7</sup>It's easy to show that term in (54) under total derivative has no  $\delta$ -function singularities.

One estimates the first term ( $x_1 = L$ ) in Eq. (58), which can be written in the form:

$$I(L) \equiv \int dp_1 e^{-ip_1 L} e^{-\frac{p_1^2}{2m^2}} H_n\left(\frac{p_1}{m}\right) \int dx_0 dx_\perp \int d\mathbf{p}_\perp f_n(\mathbf{p}_\perp) e^{i\omega x_0 - ip_\perp x_\perp} \left( \frac{1}{m} \partial_x^1 \hat{\varphi}(x) - \hat{\varphi}^1(x) \right) \quad (59)$$

By definition the integral over  $x_0, x_\perp, \mathbf{p}_\perp$  in Eq. (59) is a distribution (generalized function) of polynomial grows or Shwartz distribution, (renormalizable theory), i.e. it can grow not faster than polynomial:  $\leq C_1 p_1^l |L|^k$ , where  $l$  and  $k$  are some numbers,  $C_1$  is a constant.

Then

$$\begin{aligned} I(L) &\leq C_1 |L|^k \int dp_1 p_1^l e^{-ip_1 L} e^{-\frac{p_1^2}{2m^2}} H_n\left(\frac{p_1}{m}\right) \\ &< C_1 |L|^k \left| \left( -i \frac{\partial}{\partial L} \right)^{l+n} \right| \int dp_1 e^{-ip_1 L - \frac{p_1^2}{2m^2}} \\ &= C_1 |L|^k \left| \left( -i \frac{\partial}{\partial L} \right)^{l+n} \right| e^{-(Lm)^2} \end{aligned} \quad (60)$$

Thus  $I(L)$  decreases as  $\sim e^{-(Lm)^2} |L|^{k+l+n}$ , and we proved that the second term in Eq. (53) equals zero.

If we repeat the LSZ procedure for the second operator  $a_{out}^+(n)$  in Eq. (51) we get

$$\begin{aligned} \langle k; out | k; in \rangle &= C \langle k-2; out | \int d^4 x_1 d^4 x_2 d^3 \mathbf{p}_1 d^3 \mathbf{p}_2 e^{ip_1 x_1} f_{n_1}^*(\mathbf{p}_1) f_{n_2}(\mathbf{p}_2) \\ &\quad \left( \frac{\vec{\square}_1 + m^2}{m} \right) \left\{ e^{ip_2 x_2} \left( \frac{\vec{\square}_2 + m^2}{m} \right) T \hat{\varphi}(x_1) \hat{\varphi}(x_2) \right. \\ &\quad \left. + \partial_\mu^{x_2} [e^{ip_2 x_2} (\partial_{x_2}^\mu T \hat{\varphi}(x_1) \hat{\varphi}(x_2) - T \hat{\varphi}(x_1) \hat{\varphi}(x_2))] \right\} |k; in \rangle \end{aligned} \quad (61)$$

Again one can prove the last term under total derivative equals zero due to properties of packets  $f_n(\mathbf{q})$ . Continuing this inductive procedure we go to conclusion that all physical matrix elements of  $S$ -matrix in DKP theory coincide with that of in KGF theory independently of character interaction on these theories, if in the both theories the LSZ asymptotic conditions (50) are implemented for Heisenberg operators<sup>8</sup>.

From general result (61) the equivalence between DKP and KGF theories also follows in case interaction spin-0 particles with external eletromagnetic field,  $A_\mu^{ex}(x)$ , and gravitational fields,  $e_{\mu a}^{ex}(x)$ , see Eqs. (9) and (30)-(32).

<sup>8</sup>The quasilocal terms do not appear in Eq. (61) [15].

Now we briefly describe the proof of equivalence of the theories in case interaction spin-0 particles with non-abelian (Yang-Mills) external and quantized fields  $A_\mu^i$ , where  $i = 1, \dots, 2N^2 - 1$ , is the group index of  $SU(N)$  group.

One is restricted by case when spin-0 particles in DKP and KGF theories are taken on fundamental representation of the  $SU(N)$  group.

The initial density of Lagrangian in DKP theory is (in  $\alpha$ -gauge)

$$\begin{aligned} \mathcal{L}_{DKP} = & \bar{\psi}_\alpha(x) (i\beta_\mu D_{ab}^\mu - m\delta_{ab}) \psi_b - \frac{1}{4} F_{\mu\nu}^i F_i^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A_\mu^i)^2 \\ & - \bar{C}_a (\partial_\mu D_{ab}^\mu C_b) \end{aligned} \quad (62)$$

Here:  $(D^\mu)_{ab} = \partial^\mu \delta_{ab} - ig(A_\mu^i T_i)_{ab}$ ,  $a = 1, \dots, N$ , index of fundamental representation.  $F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + ig f^{ijk} A_\mu^j A_\nu^k$ ;  $A_\mu^i$ -is Yang-Mills field;  $T_i$  are  $N \times N$  matrixes with the commutation relation

$$[T_i, T_j] = i f_{ijk} T_k \quad (63)$$

$f_{ijk}$  are structure constant of  $SU(N)$  group;  $\bar{C}_a, C_b$  Faddeev-Popov anticommuting ghost fields. We also write down only spin-0 part of  $\mathcal{L}_{int}$  in KGF theory:

$$\mathcal{L}_{KGF} = -\varphi_a [(D_\mu)_{ab} (D^\mu)_{bc} + m^2 \delta_{ab}] \varphi_c \quad (64)$$

Now, utilizing LSZ reduction formulas we shall obtain the same formulas as Eqs. (53)-(61) excluding appearance additional group index  $a$  at all operators and at solution of free equations,  $f_{nk}^*$ , in Eqs. (51) and (52). Therefore, the proof of equivalence of the physical matrix elements of  $S$ -matrix for spin-0 particles and for GF of any number of the Yang-Milles particles can be carried out with help of the same arguments as before.

## 4 Conclusion

1. Starting from Lagrangian approach to DKP theory of spin-0 particles we constructed generating functional for many particles GF of the theory (see Eqs. (1), (3), (29)) and utilizing the LSZ reduction formulas, we strictly proved the equivalence between physical matrix elements of  $S$ -matrix in DKP and KGF theories, being the general formula (61) can be applied to any type of interaction on the both theories, and consequently to interaction spin-0 particles with quantized Maxwell, Yang Milles fields and with external gravitational fields. We also proved (see for instance Eqs. (5)-(7)) that in DKP and KGF theories the many particles GF of photons and Yang-Mills particles exactly coincide (not only physical matrix elements).
2. Considering the interaction of DKP spin-0 particles with external gravitational field we restricted by two cases: without torsion, when connection

$\Gamma_{\mu\nu}^\lambda = \overset{\circ}{\Gamma}_{\mu\nu}^\lambda$  (see Eq. (14)) is Levi-Civita one and with torsion, when the total connection  $\Gamma_{\mu\nu}^\lambda$  is Cartan connection<sup>9</sup>, Eq. (16), antisymmetrical part of which, Eq. (18), expressed through torsion tensor. This case is one of teleparallel description of gravity [20, 21, 22, 18, 19]. We would like to note that there is the third case interaction with gravitational field, which leads to Einstein-Cartan gravity [17]. In the last case the metric postulate is implemented:

$$\nabla_\mu g_{\nu\lambda} = \partial_\mu g_{\nu\lambda} - \Gamma_{\mu\nu}^\rho g_{\rho\lambda} - \Gamma_{\mu\lambda}^\rho g_{\nu\rho} = 0 \quad (65)$$

here  $\Gamma_{\mu\nu}^\rho = \overset{\circ}{\Gamma}_{\mu\nu}^\rho + {}^{(-)}\Gamma_{\mu\nu}^\rho$

where  ${}^{(-)}\Gamma_{\mu\nu}^\lambda$  is antisymmetrical part of Einstein-Cartan connection. The equation for  $\psi$  now has the view (instead(19)):

$$\left( i\beta^\mu \left( \nabla_\mu + {}^{(-)}\Gamma_{\mu\rho}^\rho \right) - m \right) \psi = 0 \quad (66)$$

The proof of the equivalence of DKP and KGF equations is the same.

3. For unstable particles we suggest the following fenomenological Lagrangian of interaction:

$$\mathcal{L}_{int} = \lambda \left( \overline{\psi}^M \beta_\mu \psi^m + \overline{\psi}^m \beta_\mu \psi^M \right) j^\mu(x) \quad (67)$$

where  $\psi^M, \psi^m$  are the DKP functions, describing accordingly  $K_L^0$  and  $\pi$  mesons with masses  $M$  and  $m$ ;  $j_\mu = (\overline{e}\gamma_5\gamma_\mu\nu) + (\overline{\nu}\gamma_5\gamma_\mu e)$ ;  $\lambda$ -fenomenological constant of point-like interaction.

Then it is easy to show in  $\lambda^2$ -approximation that the imaginary part of GF of  $K_L^0$  meson, which determines the amplitude of probability decay  $K_L^0 \rightarrow \pi^+ + e^- + \overline{\nu}$ , in the DKP theory, concides with that of in KGF theories.

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<sup>9</sup>One notes that in this case the "metric postulate"  $\nabla_\mu g_{\nu\lambda} = 0$  does not implement.

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